

AVOIDING PITFALLS IN MEASUREMENT UNCERTAINTY ANALYSIS

Benny R. Smith
Inchworm Solutions
Santa Rosa, CA

Summary: Pitfalls, both subtle and obvious, await the new or casual practitioner of measurement uncertainty analysis. This paper describes the more common mistakes that are made and gives a prescription for staying out of trouble. A worked example illustrates the preferred technique.

ISO- GUM PROCESS

The ISO *Guide to the Expression of Uncertainty in Measurement* (GUM) prescribes six steps for the analysis of a measurement:

Model the Measurement

Develop the Measurement Equation

Derive the Uncertainty Equation

Determine the Standard Uncertainties

Calculate the Combined Standard Uncertainty

Determine the Expanded Measurement Uncertainty

While these steps are designed to help analysts focus on the core of a measurement and to promote consistency in uncertainty analysis, each step offers opportunities to go astray. A mistake in one step may cascade, perhaps amplified, to subsequent steps.

MEASUREMENT EXAMPLE

Absolute Amplitude Accuracy of a Spectrum Analyzer

Specification:

$ACC \leq 0.27 \text{ dB}$, for amplitude levels between -10 dBm and -50 dBm .

To verify this specification, a two-part procedure is followed. The equipment connections for each part are shown below. In Part 1, an accurate reference power level is established with a power sensor. The spectrum analyzer measures this reference level. In Part 2, several amplitude-test levels are established with a precision step-attenuator. At each test level, the spectrum analyzer

again measures the absolute amplitude. The difference between the true power available from the network and the power indicated by the spectrum analyzer is defined as the absolute amplitude accuracy of the spectrum analyzer.

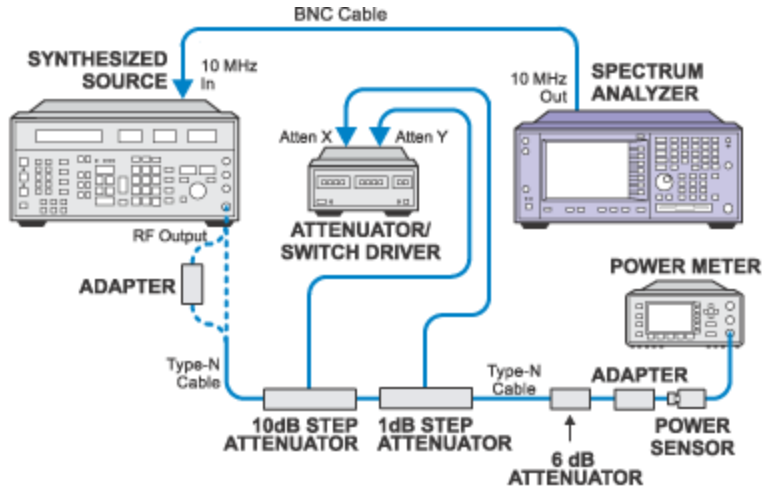


Figure 1: Measure the reference power with a power meter.

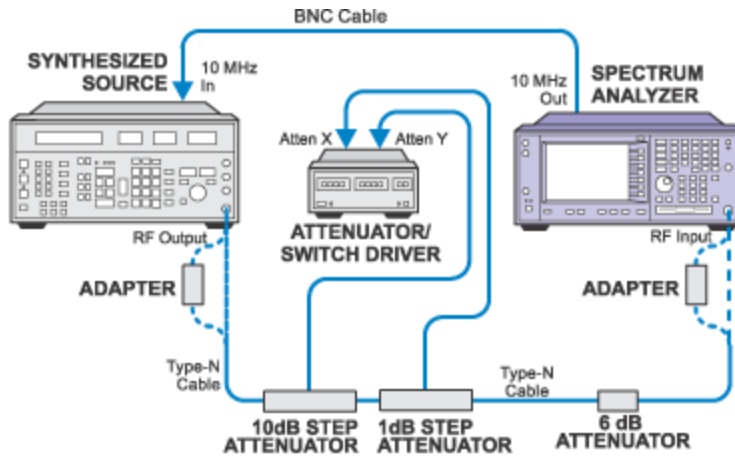
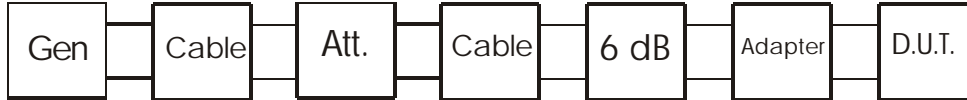


Figure 2: Measure the reference and other power levels with the spectrum analyzer.

MODEL THE MEASUREMENT

The measurement can be modeled as:



Each piece of equipment used in the measurement is modeled as either a two-port network or a one-port network. The two-ports can be represented by s-parameters: s_{11} , s_{12} , s_{21} , and s_{22} . The signal generator can be represented as an active one-port with a reflection coefficient. The spectrum analyzer and the power sensor can be represented as passive one-ports with only a reflection coefficient.

PITFALL: Omitting from the model any piece of equipment that seems insignificant.

It is unwise to eliminate anything from the model which is present in the physical network, including cables and adapters, until a measurement equation has been created and an uncertainty equation is derived. Uncertainty terms can always be set to zero later in order to check the limits of changes to the network.

MEASUREMENT EQUATION

PITFALL: Not writing a measurement equation at all.

Do not be tempted to bypass the writing of a measurement equation and go straight to an assessment of the uncertainty.

Without a measurement equation you risk:

Not identifying all of the sources of uncertainty in the measurement.

Not knowing the sensitivity of the total uncertainty to the individual contributors.

PITFALL: Treating uncertainty as a random variable separate from the associated parameter and including the uncertainty in the measurement equation.

EXAMPLE: $ACC = [P_{SOURCE} + \Delta P_{SOURCE}] - [P_{MEAS} + \Delta P_{MEAS}]$

Remember that each variable in the measurement equation is considered to be a random variable with a known (or assumed) probability distribution, characterized at the least by a mean value and a standard deviation. Adding extra terms to represent the uncertainty (standard deviation) of the variable is redundant and will lead to trouble.

PITFALL: Writing the measurement equation in the wrong domain (log or linear).

If the performance specification being verified is expressed in dB units, then the measurement equation must be written in log terms. If it is expressed in linear units (including percent), the measurement equation must be written in linear terms.

Once the measurement equation is written, it can be transformed from one domain to the other (see example below) in order to expand the original equation into a more detailed version that accounts for all of the influence parameters (cable loss, attenuator settings, reflection coefficients, etc.)

NOTE: There are two good reasons why the measurement equation should be transformed to the linear domain (if it is not written there in the first place):

In the linear domain, the measurement equation will often have the form $Y = \frac{A \times B \times C}{D \times E \times F}$, which will allow writing the uncertainty equation by inspection. (See example below.)

The terms in the uncertainty equation will have the form $\frac{U(A)}{A}$, which has the dimensions of dB or %. Many instrument specifications are quoted in the form of dB or %.

PITFALL: Incomplete or incorrect measurement equations.

Once the mathematical model (cascade of two-ports, in this example) is complete, the development of the measurement equation, while messy, should be straightforward.

EXAMPLE:

Specification: Amplitude Accuracy @ 50 MHz: $\leq 0.27 \text{ dB}$, for input signals from -10 dBm to -50 dBm.

Since the Amplitude Accuracy specification is given in dB, the measurement equation will be expressed in dB.

$$ACC_{dB} = P_{AV} - P_m \quad \text{EQN. 1}$$

where: ACC is the absolute amplitude accuracy

P_{AV} is the absolute power available from the "effective" source

P_m is the power *indicated* by the spectrum analyzer

This equation simply states that amplitude accuracy is the difference between the true power available at the spectrum analyzer's input and the power indicated by the spectrum analyzer.

The expression for P_{AV} is given by a signal-flow-graph analysis of Figure 3.

$$P_{AV} = \frac{\left[P_{gZ_0} \right]_G A_{C1} A_{ATT} A_6 A_{C2} A_A}{M_{C1} M_{ATT} M_{C2} M_6 M_A} \quad \text{EQN. 2}$$

where:

$\left[P_{gZ_0} \right]_G$ is the power available from the signal generator.

A_{ATT}, A_6 are the gains of the external attenuators.

A_{C1}, A_{C2}, A_A are the gains of the two Type N cables and the adapter.

$M_{C1} = \left| 1 - \Gamma_g s_{11a} \right|^2$ is the mismatch between the source and the first Type N cable.

$M_{ATT} = \left| 1 - \Gamma_a s_{11b} \right|^2$ is the mismatch between the first Type N cable and the composite attenuator.

$M_{C2} = \left| 1 - \Gamma_b s_{11c} \right|^2$ is the mismatch between the composite attenuator and the second Type N cable.

$M_6 = \left| 1 - \Gamma_c s_{11d} \right|^2$ is the mismatch between the second Type N cable and the 6 dB attenuator.

$M_A = \left| 1 - \Gamma_d s_{11e} \right|^2$ is the mismatch between the 6 dB attenuator and the adapter.

Notice that the expression for P_{AV} is in the linear domain, whereas the measurement equation is in the log domain. Rather than convert the expression for P_{AV} to the log domain, the measurement equation will be converted to the linear domain:

$$ACC_{LIN} = \frac{P_{AV}}{P_m} \quad \text{EQN. 3}$$

(Subtraction in the log domain is equivalent to division in the linear domain.)

For the reference amplitude level of this measurement (-10 dBm), the power sensor determines P_{AV} , the available power. The spectrum analyzer makes its own reading of the reference power.

Therefore, the measurement equation at the reference amplitude is:

$$ACC_{REF} = \frac{[P_{gZ0}]_{PS}}{P_m} = \frac{P_{AV-REF}}{P_{m-REF}} \quad \text{EQN. 4}$$

where $[P_{gZ0}]_{PS}$ is the power that the network would deliver to a matched load, as estimated by the power sensor/power meter.

In terms of the signal source and the intervening network, the measurement equation for the reference level measurement is:

$$ACC_{REF} = \frac{[P_{gZ0}]_G}{P_{m-REF}} \times \frac{A_{C1}A_{ATT}A_6A_{C2}}{M_{C1}M_{ATT}M_{C2}M_6}$$

At the test power levels:

$$ACC_{TEST} = \frac{P_{AV-TEST}}{P_{m-TEST}}$$

$$\text{Since } P_{AV-REF} = [P_{gZ0}]_G \times T_{REF} \text{ and } P_{AV-TEST} = [P_{gZ0}]_G \times T_{TEST},$$

$$P_{AV-TEST} = P_{AV-REF} \times \frac{T_{TEST}}{T_{REF}} \quad \text{EQN. 5}$$

$$\text{where: } T_{REF} = \left[\frac{A_{C1}A_{ATT}A_6A_{C2}}{M_{C1}M_{ATT}M_{C2}M_6} \right]_{REF} \text{ and } T_{TEST} = \left[\frac{A_{C1}A_{ATT}A_6A_{C2}A_A}{M_{C1}M_{ATT}M_{C2}M_6M_A} \right]_{TEST}$$

Therefore:

$$ACC_{TEST} = \frac{P_{AV-REF}}{P_{m-TEST}} \times \frac{\left[\frac{A_{C1}A_{ATT}A_6A_{C2}}{M_{C1}M_{ATT}M_{C2}M_6} \right]_{TEST}}{\left[\frac{A_{C1}A_{ATT}A_6A_{C2}A_A}{M_{C1}M_{ATT}M_{C2}M_6M_A} \right]_{REF}} \quad \text{EQN. 6}$$

NOTE: Since the final coaxial adapter has been deleted for the test connection to the spectrum analyzer, the terms A_A and M_A do not appear in EQN. 6. (If an adapter is needed, the terms can be re-inserted into T_{TEST} .)

Since A_{C1}, A_{C2} and A_6 do not change throughout the test,

$$ACC_{TEST} = \frac{P_{AV-REF}}{P_{m-TEST}} \times \frac{\left[\frac{A_{ATT}}{M_{C1} M_{ATT} M_{C2} M_6 M_A} \right]_{TEST}}{\left[\frac{A_{ATT} A_A}{M_{C1} M_{ATT} M_{C2} M_6 M_A} \right]_{REF}} \quad \text{EQN. 7}$$

Define: $[A_{ATT}]_{TEST} = A_{TEST}$

$$[A_{ATT} \cdot A_A]_{REF} = A_{REF}$$

$$[M_{C1} M_{ATT} M_{C2} M_6]_{TEST} = M_{TEST}$$

$$[M_{C1} M_{ATT} M_{C2} M_6 M_A]_{REF} = M_{REF}$$

Then: $ACC_{TEST} = \frac{[P_{AV}]_{PS}}{P_m} \times \frac{A_{TEST}}{A_{REF}} \times \frac{M_{REF}}{M_{TEST}} \quad \text{EQN. 8}$

UNCERTAINTY EQUATION

The classic ISO-GUM uncertainty equation is:

$$U^2(ACC) = C_{PS}^2 \times U^2(P_{AV-PS}) + C_m^2 \times U^2(P_m) + C_{AT}^2 \times U^2(A_{TEST}) \\ + C_{AR}^2 \times U^2(A_{REF}) + C_{MT}^2 \times U^2(M_{TEST}) + C_{MR}^2 \times U^2(M_{REF})$$

where: $C_{AT} = \frac{\partial ACC}{\partial A_{TEST}}$, $C_m = \frac{\partial ACC}{\partial P_m}$, etc.

PITFALL: Partial differentiation in determining sensitivity coefficients.

It is possible to avoid partial differentiation if the measurement equation has the form:

$$Y = \frac{A \times B \times C}{D \times E \times F}$$

The uncertainty equation can be written as:

$$\frac{U_C^2(ACC)}{ACC^2} = \frac{U^2(P_{REF})}{(P_{REF})^2} + \frac{U^2(P_m)}{(P_m)^2} + \frac{U^2(A_{TEST})}{(A_{TEST})^2} + \frac{U^2(A_{REF})}{(A_{REF})^2} + \frac{U^2(M_{TEST})}{(M_{TEST})^2} + \frac{U^2(M_{REF})}{(M_{REF})^2}$$

This equation contains *relative* uncertainty terms, each of which is dimensionless. This special form of the general ISO-GUM uncertainty equation is very handy for many uncertainty analysis situations.

STANDARD UNCERTAINTY

PITFALL: Confusing Type A and Type B uncertainty.

Type A evaluation of measurement uncertainty requires the taking of data from repeated observations of the subject parameter. For example, the DC voltage accuracy of a particular model of DVM could be established by measuring a voltage standard repeatedly, under the same circumstances, with several DVM units. The mean and standard deviation of the resulting data would provide a nominal value (mean) and an uncertainty (standard deviation) of that value.

Type B evaluation of measurement uncertainty is obtained by applying scientific judgment to *all* of the available information about the parameter in question, including manufacturer's data sheets, previous measurement data, etc.

NOTE: In common verification testing using off-the-shelf test equipment and standards, Type B evaluation of uncertainty is prevalent.

EXAMPLE: Type A Evaluation of uncertainty

From the uncertainty equation, evaluate the standard relative uncertainty: $\frac{U(A_{TEST})}{(A_{TEST})}$.

The relative uncertainty of the attenuation value of the composite attenuator is taken from the calibration report for the attenuator. The example data shown below is typical of that obtained from a standards laboratory. Assume that the calibration data has a gaussian probability distribution and that the values given represent the 2s (or 95%) confidence limits.

Precision Attenuator Accuracy Data at 50 MHz

Composite Atten. Setting (dB)	Atten. from Cal. Report (dB)	Atten. Cal. Uncertainty (± dB)
10	9.959	0.005
12	11.917	0.005
20	19.987	0.005
25	59.650	0.005
35	34.960	0.005
50	49.773	0.005

$$s_{CAL} = \frac{\text{Cal. Unc. (dB)}}{2} = \pm \frac{0.005 \text{ dB}}{2} = \pm 0.0025 \text{ dB}$$

Converting to the linear domain (since the uncertainty equation is written in the linear domain):

$$\left[\frac{U(A_{ATT})}{A_{ATT}} \right] = \left[10^{\frac{-0.025}{10}} - 1 \right] = \pm 0.0006$$

PITFALL: How to determine the underlying distribution and the confidence level for a performance parameter value taken from an instrument's data sheet.

EXAMPLE: Type B Evaluation of uncertainty

The true power delivered to the spectrum analyzer is estimated by the power meter. The equation that describes the available power in terms of the power meter and power sensor parameters is:

$$P_{AV} = \frac{M_{PS} P_m}{IKLm}$$

where P_m is the power indicated by the power meter.

The relative uncertainty of this parameter is: $\frac{U(P_m)}{P_m}$

The power meter reading is a digital value, delivered via the front-panel display or via the GPIB. The resolution of this reading is selectable. Assume for this example, that the resolution is 0.1 dB. Any power reading that falls within one of the 0.1 dB intervals will be given the same reported value. Therefore, a reported value could be due to any value within the interval. This situation is best modeled by a uniform distribution.

For a uniform distribution of width 0.1 dB, the standard deviation is:

$$s_{SCALE} = \frac{0.1 \text{ dB}}{\sqrt{12}} = 0.03 \text{ dB}.$$

Converting to linear terms:

$$\left[\frac{U(P_m)}{P_m} \right] = \left[10^{\frac{\pm 0.03}{10}} - 1 \right] = \pm 0.007$$

NOTE: For most specifications given in manufacturer's data sheets, assume that the value given represents at least 95% of the population of values. 95% is equivalent to 2s for a gaussian (normal) distribution and 1.645s for a uniform distribution. Choose an assumed distribution based on the nature of the specification. Most specifications can be treated as if they are taken from a gaussian distribution.

PITFALL:

Do data sheet values represent $U(A)$ or $\frac{U(A)}{A}$?

$U(A)$ will always have dimensional units: volts, ohms, hertz, etc.

$\frac{U(A)}{A}$ will always be dimensionless. It will be specified as %, dB, or as a ratio.

COMBINED STANDARD UNCERTAINTY

Once a value is obtained for each standard uncertainty represented in the uncertainty equation, the combined standard uncertainty can be found.

NOTE: The uncertainty equation has the form:

$$s_C^2 = s_A^2 + s_B^2 + s_C^2 + s_D^2 + \dots \quad \text{EQN.9}$$

Each of the squared terms in this equation represents a statistical variance. Standard deviation (a.k.a, standard uncertainty) is defined as the square-root of the variance, regardless of the statistical distribution involved.

EQN. 9 states that the combined variance of the response parameter (absolute amplitude accuracy, in this example) is equal to the sum of the individual variances of the contributing parameters.

Recalling the uncertainty equation:

$$\frac{U_c^2(ACC)}{ACC^2} = \frac{U^2(P_{REF})}{(P_{REF})^2} + \frac{U^2(P_m)}{(P_m)^2} + \frac{U^2(A_{TEST})}{(A_{TEST})^2} + \frac{U^2(A_{REF})}{(A_{REF})^2} + \frac{U^2(M_{TEST})}{(M_{TEST})^2} + \frac{U^2(M_{REF})}{(M_{REF})^2}$$

PITFALL: The units of each standard-uncertainty term are not dimensionless, or are not consistent with one another.

Since the uncertainty equation is written in the linear domain, each standard-uncertainty term (e.g., $\frac{U(M_{REF})}{M_{REF}}$) will be expressed as a decimal number or as a %. Clearly, the terms must all have the same units: either decimal or %.

The combined standard uncertainty, $\frac{U_c(ACC)}{ACC}$, represents the standard deviation of the composite random variable, ACC . If there are three or more contributors to $\frac{U_c(ACC)}{ACC}$, then the resulting distribution will be approximately gaussian, regardless of the distributions of the contributors.

EXPANDED UNCERTAINTY

$\frac{U_c(ACC)}{ACC}$ is a linear quantity. The original specification for Absolute Amplitude Accuracy is a logarithmic quantity (dB). Converting $\frac{U_c(ACC)}{ACC}$ back to the log domain gives $U(ACC)$ in dB.

EXAMPLE:

$$\frac{U_c(ACC)}{ACC} = \pm 0.012$$

$$U_c(ACC)_{dB} = 10 \times \log \left(1 \pm \frac{U_c(ACC)}{ACC} \right) = 10 \times \log(1 \pm 0.012) \cong \pm 0.052 \text{ dB}$$

$$U_c(ACC)_{dB} = s_{ACC} = \pm 0.052 \text{ dB}$$

PITFALL: Choosing the correct “coverage factor” for the final uncertainty value.

Now that the combined standard uncertainty is back in the log domain, it is a simple matter to determine a “coverage” factor that, when multiplied by the combined standard uncertainty, will give the desired degree of confidence that the true value of Absolute Amplitude Accuracy is contained within the limits $ACC_{dB} \pm U_c(ACC_{dB})$.

For a gaussian distribution, a coverage factor of 2 will provide 95% coverage (i.e., $2s$).

The expanded uncertainty becomes: $2 \times U_c(ACC_{dB})$.

$$U_{EXP}(ACC_{dB}) = \pm 0.10 \text{ dB}$$